New Simple Proofs of the Two-Port Stability Criterium in Terms of the Single Stability Parameter μ_1 (μ_2)

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***Abstract*—The classical scattering-parameter stability criterium for a linear two-port makes use of two conditions involving the Rollet parameter K plus one additional parameter. A new stability criterium was developed by Edwards and Sinksky on the basis of a condition on a single parameter, i.e., μ_1 or μ_2. This paper presents a new, simpler, and more straightforward set of proofs of the single-parameter stability criterium for a linear two-port. The first proof is algebraic and shows the equivalence of the conditions $K > 1$, $b_i > 1$ with the condition $\mu_i > 1$ ($i = 1, 2$). The second proof, which is geometrical, relies only on the classical stability circle concepts in an improved way with respect to the treatment by Edwards and Sinksky.**

***Index Terms*—Linear circuits, scattering parameters, stability criteria, two-port circuits.**

I. Introduction

ALMOST 40 years after Rollet's paper [2] was published, the classical problem of the stability criterium for a linear two-port still seems to attract the attention of the microwave and RF community, as a source of practically significant, but also theoretically, appealing analytical investigations. The aim of this paper is to add some new elements to the existing body of knowledge by providing a new set of simple straightforward analytical and geometrical proofs for the so-called single-parameter stability criterion [1].

A well-established result allows the unconditional two-port stability to be expressed in terms of the two-port scattering parameters in different, albeit equivalent, ways, all having in common the condition of the Rollet parameter K [2]

$$K = \frac{1 - |s_{11}|^2 - |s_{22}|^2 + |\Delta|^2}{2|s_{12}s_{21}|} > 1 \tag{1}$$

together with *one* of the following conditions:

$$|\Delta| = |s_{11}s_{22} - s_{12}s_{21}| < 1 \tag{2}$$

$$b_1 = \frac{1 - |s_{11}|^2}{|s_{12}||s_{21}|} > 1 \tag{3}$$

$$B_1 = 1 + |s_{11}|^2 - |s_{22}|^2 - |\Delta|^2 > 0 \tag{4}$$

$$b_2 = \frac{1 - |s_{22}|^2}{|s_{12}||s_{21}|} > 1 \tag{5}$$

$$B_2 = 1 + |s_{22}|^2 - |s_{11}|^2 - |\Delta|^2 > 0. \tag{6}$$

Notice that, while K is independent from the scattering-parameter normalization, all the other parameters depend on the port normalization impedances, although the above conditions are independent from them. If $K > 1$, the above auxiliary conditions are all equivalent, as was proven in [3] for what concerns the conditions (2), (4), and (6), and in [1] for what concerns (3) and (5). Since (3) and (4) are relative to stability at port 1, while (5) and (6) refer to stability at port 2, this also leads to the well-known result that stability at one port implies global two-port stability.

The classical stability criterion, as summarized above, was, and still is, widely exploited in computer-aided design (CAD) tools; it is, however, somewhat inconvenient since the simple condition $K > 1$ is necessary, but not sufficient to ensure stability. In 1992, Edwards and Sinksky [1] proved that a condition on a single parameter is sufficient to assess unconditional stability; namely, they showed that the two-port unconditional stability conditions can be put into biunique correspondence with one of the following:

$$\mu_1 = \frac{1 - |s_{11}|^2}{|s_{22} - s_{11}^*\Delta| + |s_{12}s_{21}|} > 1 \tag{7}$$

$$\mu_2 = \frac{1 - |s_{22}|^2}{|s_{11} - s_{22}^*\Delta| + |s_{12}s_{21}|} > 1. \tag{8}$$

Despite the obvious advantages of using a single stability parameter, and although the proof in [1] has never been questioned, as far as the authors' knowledge, the popularity of the new $\mu_{1,2} > 1$ criterium among RF and microwave designers was perhaps lower that expected.[1] A possible reason is perhaps the fact that the original proof in [1] is quite involved since it is based on Γ_G and Γ_L stability circles, thus implying a cumbersome analysis of multiple cases (generator and load reflection coefficients mapped inside or outside those circles) and longer calculations. The proof in [1] finally concludes that (7) implies (1) and (5), i.e., unconditional stability.

In this paper, we propose a much more straightforward treatment of stability conditions (7) and (8). In fact, in Section II, we show that the equivalence between (7) and (8) and conditions (1) and (3) [see (1) and (5)] can be proven through an extremely simple and direct algebraic technique. Section II also includes the discussion of a new relationship between μ, K, and b, which is introductorily exploited to provide a direct "by inspection" proof of the equivalence between the above conditions. Section III, on the other hand, proposes a much simpler variation

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[1] As a matter of fact, most of the more recent CAD tools do include the new stability criterion together with the old ones.

of the original "geometrical" demonstration proposed in [1], in which the discussion is based on the $\Gamma_{\rm OUT}$ ($\Gamma_{\rm IN}$) circles instead of Γ_L (Γ_G) ones.

II. New Algebraic Proof of the Single-Parameter Criterium

A. Introductory Relationship Between μ, K, and b

In this section, we will establish a relationship between μ_1 and the classical stability parameters K and b_1. We start from the definition of μ_1, which can be written as

$$|s_{22} - s_{11}^*\Delta| = \frac{1-|s_{11}|^2}{\mu_1} - |s_{12}s_{21}|. \quad (9)$$

Since

$$|s_{22} - s_{11}^*\Delta|^2 = \left(|s_{22}|^2 - |\Delta|^2\right)\left(1-|s_{11}|^2\right) + |s_{12}|^2|s_{21}|^2 \quad (10)$$

squaring both terms of (9), subtracting on both sides the factor $|s_{12}s_{21}|^2$, and using (10), we have

$$\left(|s_{22}|^2 - |\Delta|^2\right)\left(1-|s_{11}|^2\right) = \frac{\left(1-|s_{11}|^2\right)^2}{\mu_1^2} - 2|s_{21}||s_{12}|\frac{1-|s_{11}|^2}{\mu_1}$$

and, therefore,

$$|s_{22}|^2 - |\Delta|^2 = \frac{1-|s_{11}|^2}{\mu_1^2} - 2\frac{|s_{21}||s_{12}|}{\mu_1}. \quad (11)$$

Since

$$|s_{22}|^2 - |\Delta|^2 = -|s_{21}||s_{12}|(2K - b_1)$$

after some rearrangement, (11) gives

$$\frac{b_1}{\mu_1^2} - \frac{2}{\mu_1} + 2K - b_1 = 0. \quad (12)$$

This equation always has real solutions since its discriminant is positive as follows:

$$1 - b_1(2K - b_1) = \frac{|s_{22} - s_{11}^*\Delta|^2}{|s_{12}|^2|s_{21}|^2} > 0$$

moreover, rearranging (9), one obtains

$$\frac{b_1}{\mu_1} = \frac{|s_{22} - s_{11}^*\Delta|}{|s_{12}s_{21}|} + 1 > 1$$

thus implying that only the solution of (12) with the plus sign is valid as follows:

$$\mu_1 = \frac{b_1}{1+\sqrt{1-2Kb_1+b_1^2}}. \quad (13)$$

A direct contour plot of μ_1 as a function of K and b_1 from (13) shows (see Fig. 1) that the region with $K > 1$ and $b_1 > 1$, where μ_1 is real, corresponds to $\mu_1 > 1$, and *vice versa* (i.e., $\mu_1 < 1$ anywhere else) [1]. (The same conclusions can obviously be shown to hold for the set μ_2, K, and b_2.) A straightforward algebraic proof of the above equivalence is provided in Section II-B.

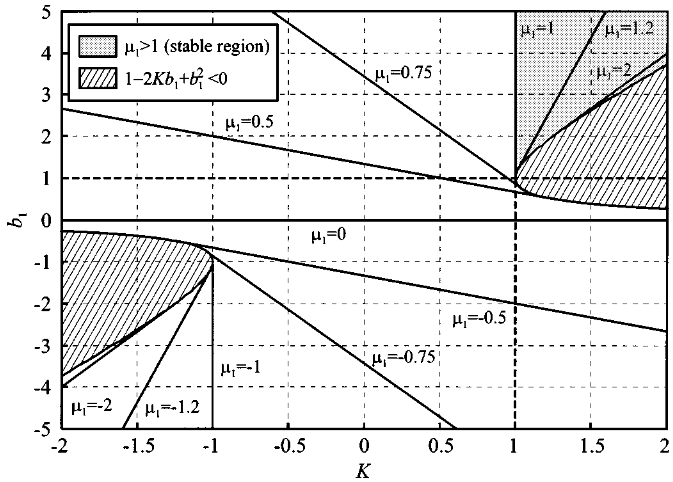

Fig. 1. Behavior of the μ_1 stability parameter in the plane K, b_1. The shaded areas are bounded by two hyperbolae defined by $1 - 2Kb_1 + b_1^2 = 0$, i.e., by the locus where the discriminant of (12) vanishes.

B. Algebraic Proof of the Stability Criterium

We will now give a simple demonstration of the equivalence, which has been shown to hold through graphical inspection, of the two conditions (1) and (3) with (7).

We first notice that, from (7), it is trivial to see that $\mu_1 < 0 < 1$ implies $|s_{11}| > 1$ and *vice versa* so that when the two-port cannot be unconditionally stable because $|s_{11}| > 1$, the parameter μ_1 correctly predicts conditional stability. We can, therefore, focus on the case $|s_{11}| < 1$ or $\mu_1 > 0$.

We now prove that μ_1 implies $K > 1$ and $b_1 > 1$. In fact, condition $\mu_1 > 1$ can be expressed as

$$|s_{22} - s_{11}^*\Delta| < 1 - |s_{11}|^2 - |s_{12}s_{21}| = |s_{21}||s_{12}|(b_1 - 1) \quad (14)$$

from which we immediately find that $b_1 > 1$. Thus, by squaring (14), and using (10), we then find that

$$\left(|s_{22}|^2 - |\Delta|^2\right)\left(1-|s_{11}|^2\right) + |s_{12}s_{21}|^2\left(1-|s_{11}|^2\right)^2 - 2|s_{12}s_{21}|\left(1-|s_{11}|^2\right) + |s_{12}s_{21}|^2. \quad (15)$$

Since $|s_{11}| < 1$ and, therefore, $1 - |s_{11}|^2 > 0$, it follows that

$$|s_{22}|^2 - |\Delta|^2 < 1 - |s_{11}|^2 - 2|s_{12}s_{21}| \quad (16)$$

and, therefore,

$$1 - |s_{11}|^2 - |s_{22}|^2 + |\Delta|^2 > 2|s_{12}s_{21}| \quad (17)$$

i.e., from the definition of the Rollet parameter $K > 1$.

By reversing the above proof, we now show that $b_1 > 1$ and $K > 1$ implies $\mu_1 > 1$. Starting from (17), we obtain (16) and then (15) by multiplying (16) by the factor $1 - |s_{11}|^2$ (positive because $b_1 > 1$), and adding $|s_{12}s_{21}|^2$ to both sides. Taking the square root of the two terms, and using (10), we have

$$|s_{22} - s_{11}^*\Delta| \le \left|1 - |s_{11}|^2 - |s_{12}s_{21}|\right|$$

but since $b_1 > 1$, we have $1 - |s_{11}|^2 - |s_{12}s_{21}| > 0$ and, finally, we obtain (14), which is equivalent to condition $\mu_1 > 1$.

By means of the same procedure, we find that $\mu_2 > 1$ implies $K > 1$, $b_2 > 1$ and *vice versa*. The reciprocal implication of the conditions $\mu_1 > 1$ and $\mu_2 > 1$ can be stated equivalently by showing that the condition $K > 1$, $b_1 > 1$ implies $b_2 > 1$. This demonstration has already been carried out in [1], but a simpler proof will be given here, which also has the advantage of immediately showing that if $K > 1$, conditions $b_1 > 1$ and $b_2 > 1$ mutually imply each other.

We start by evaluating the product $(b_1 - 1)(b_2 - 1)$; a direct computation shows that

$$\begin{aligned}(b_1 - 1)(b_2 - 1) &= \frac{1 - |s_{11}|^2 - |s_{22}|^2 + |s_{11}|^2|s_{22}|^2}{|s_{12}|^2|s_{21}|^2} \\ &\quad + \frac{|s_{21}|^2|s_{12}|^2 - 2|s_{21}||s_{12}| + |s_{21}||s_{12}|\left(|s_{11}|^2 + |s_{22}|^2\right)}{|s_{12}|^2|s_{21}|^2} \\ &= \frac{1 - |s_{11}|^2 - |s_{22}|^2 + |\Delta|^2 - 2|s_{21}||s_{12}|}{|s_{12}|^2|s_{21}|^2} \\ &\quad + \frac{|s_{21}||s_{12}|\left(|s_{11}|^2 + |s_{22}|^2\right) + 2\Re\left(s_{11}s_{22}s_{21}^*s_{12}^*\right)}{|s_{12}|^2|s_{21}|^2} \\ &= 2\frac{K - 1}{|s_{21}||s_{12}|} + \frac{|s_{11}\sqrt{s_{21}^*s_{12}^*} + s_{22}^*\sqrt{s_{21}s_{12}}|^2}{|s_{12}|^2|s_{21}|^2}.\end{aligned}$$

This implies that if $K > 1$, $(b_1 - 1)(b_2 - 1) > 0$, i.e., b_1 and b_2 are both either larger or smaller than unity. Therefore, condition $K > 1$, $b_1 > 1$ implies $b_2 > 1$ and, thus, $\mu_1 > 1$ implies $\mu_2 > 1$.

III. Improved "Geometrical" Proof of the Single-Parameter Criterium

Geometrical demonstrations based on concepts like input and output stability circles have a greater appeal than purely algebraic proofs owing to their more straightforward interpretation. In this section, Γ_G and Γ_L are the generator and load reflection coefficients and $\Gamma_{\rm IN}$ and $\Gamma_{\rm OUT}$ are the reflection coefficients seen at the two-port input and output. As is well known, a bilinear transformation relates the load, input, generator, output coefficients, and the scattering parameters; thus, one has

$$\Gamma_{\rm IN} = s_{11} + \frac{s_{12}s_{21}\Gamma_L}{1 - s_{22}\Gamma_L}$$

and

$$\Gamma_{\rm OUT} = s_{22} + \frac{s_{12}s_{21}\Gamma_G}{1 - s_{11}\Gamma_G}.$$

We start from the stability parameter μ_1, whose expression (7) can be revisited as

$$\mu_1^{-1} = \frac{|s_{22} - s_{11}^*\Delta| + |s_{12}s_{21}|}{1 - |s_{11}|^2}.$$

A direct relationship can be established between the stability parameter and the radius and center of the image, in the $\Gamma_{\rm OUT}$ plane, of the circle $|\Gamma_G| = 1$; as is well known, this image is a circle of center $C_{\rm OUT}$ and radius $r_{\rm OUT}$ given by

$$C_{\rm OUT} = \frac{s_{22} - s_{11}^*\Delta}{1 - |s_{11}|^2} \quad (18)$$

$$r_{\rm OUT} = \frac{|s_{12}s_{21}|}{\left|1 - |s_{11}|^2\right|}. \quad (19)$$

Using these expressions, and excluding the case $\mu_1 < 0$ or $|s_{11}| > 1$, we obtain a relationship similar to the one derived in [1]

$$\mu_1^{-1} = |C_{\rm OUT}| + r_{\rm OUT}.$$

A necessary condition for unconditional stability is

$$|C_{\rm OUT}| + r_{\rm OUT} < 1 \quad (20)$$

or, in other words,

$$\mu_1 > 1.$$

Otherwise, in fact, the image of $|\Gamma_G| = 1$, which is given by

$$\Gamma_{\rm OUT} = C_{\rm OUT} + r_{\rm OUT}e^{j\theta}, \qquad 0 \le \theta < 2\pi \quad (21)$$

would either cross the $\Gamma_{\rm OUT}$ Smith chart unit circle or be external to it, thus implying that some passive generator coefficient $|\Gamma_G| < 1$ exists that map into $|\Gamma_{\rm OUT}| > 1$.

In order to prove that this is also a sufficient condition for unconditional stability at the input port, we have to show that $|\Gamma_G| < 1$ maps into the *interior* of the circle of center $C_{\rm OUT}$ and radius $r_{\rm OUT}$. If this condition and the one given by (20) are true, $|\Gamma_{\rm OUT}| < 1$ whenever $|\Gamma_G| < 1$. Since $\Gamma_G = 0$ satisfies $|\Gamma_G| < 1$, the demonstration is complete if we can prove that the corresponding point $\Gamma_{\rm OUT} = s_{22}$ belongs to the *interior* of the circle given by (21). This is equivalent to stating that

$$|C_{\rm OUT} - s_{22}| < r_{\rm OUT}. \quad (22)$$

Using (18) and (19), remembering that the only nontrivial case is $|s_{11}| < 1$, (22) can be so rearranged as follows:

$$\left|s_{22} - s_{11}^*\Delta - s_{22}\left(1 - |s_{11}|^2\right)\right| < |s_{12}s_{21}|.$$

With some trivial manipulations, this gives

$$\begin{aligned}&\left|s_{22} - s_{11}^*\Delta - s_{22} + s_{22}|s_{11}|^2\right| \\ &\quad = \left|s_{22} - |s_{11}|^2s_{22} + s_{11}^*s_{12}s_{21} - s_{22} + s_{22}|s_{11}|^2\right| \\ &\quad = |s_{11}||s_{12}s_{21}| < |s_{12}s_{21}|\end{aligned}$$

which is surely true since $|s_{11}| < 1$.

A similar demonstration can be performed for the output stability parameter μ_2. Since

$$C_{\rm IN} = \frac{s_{11} - s_{22}^* \Delta}{1 - |s_{22}|^2}$$
$$r_{\rm IN} = \frac{|s_{12}s_{21}|}{\left|1 - |s_{22}|^2\right|}$$

where $C_{\rm IN}$ and $r_{\rm IN}$ are the center and radius of the image, in the $\Gamma_{\rm IN}$ plane, of the circle $|\Gamma_L| = 1$, we have

$$\mu_2^{-1} = |C_{\rm IN}| + r_{\rm IN}$$

and $\mu_2 > 1$ is equivalent to

$$|C_{\rm IN}| + r_{\rm IN} < 1$$

which is a necessary condition for unconditional stability at the output port. In a way similar to (22), it can be proven that

$$|C_{\rm IN} - s_{11}| < r_{\rm IN}$$

meaning that the interior of the circle $|\Gamma_L| < 1$ is always mapped, excluding the trivial case $|s_{22}| > 1$, within the unit circle $|\Gamma_{\rm IN}| < 1$. It is, therefore, proven that $\mu_2 > 1$ is a sufficient condition for the output unconditional stability.

The last thing to be proven is the reciprocal implication of the unconditional stability on the input and output ports. This demonstration, already developed from a mathematical point-of-view in Section II, has been carried out in [4] beginning from geometrical and physical concepts. The proof in [4], which we only summarize here, first shows that, given a Γ_L and Γ_G such that $\Gamma_G\Gamma_{\rm IN}(\Gamma_L) = 1$, then $\Gamma_L\Gamma_{\rm OUT}(\Gamma_G) = 1$. In this way, it is proven *ab absurdo* that the existence of a Γ_L coefficient with $|\Gamma_L| < 1$ and $|\Gamma_{\rm IN}| > 1$ implies the existence of a Γ_G with $|\Gamma_G| < 1$ and $|\Gamma_{\rm OUT}| > 1$ and *vice versa*.

IV. Conclusions

In this paper, an improved set of proofs has been provided for the single-parameter stability criterium based on the equivalent conditions $\mu_1 > 1$ ($\mu_2 > 1$). Two alternative demonstrations, the first one algebraic, the second one geometrical, have also been provided in a simpler and more straightforward manner than the one originally provided in [1], in which the single-parameter criterium was originally proposed.

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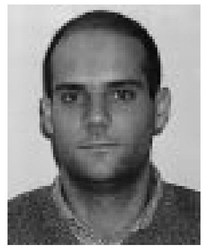

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